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GAS FLOWS WITH SPIRAL AND HELICAL LEVEL LINES

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An invariant submodel of rank 1 describing flows with spiral and helical level lines is considered. It is shown that, for the case of spiral level lines, there is a smooth gas flow in a twisted Laval nozzle. For the case of helical level lines, two flows are conjugated via the helical surface of the shock wave.

Introduction. Invariant solutions of gas-dynamic equations are considered within the framework of the SUBMODELS program [1]. By now, a classification of submodels has been carried out, general properties have been identified, and individual properties of some submodels have been studied [2, 3]. The invariant solution of rank 1 is determined by a three-dimensional subalgebra admissible by gas-dynamic equations. Let us characterize a set of such solutions as a whole. There are 37 subalgebras for the general equation of state, generating invariant submodels of rank 1. Invariant submodels are systems of ordinary differential equations, most of which can be integrated. Nonintegrable submodels, eight of which are autonomous systems and three are nonautonomous, are reduced to a system of two first-order equations. One of the examined nonautonomous systems (subalgebra 3.1 in [1, Table 6]) describes stationary conical flows [4, p. 318], two others are not sufficiently studied (subalgebras 3.3 and 3.4). Autonomous systems are reduced to the well-known equations: Riccati equation (subalgebras 3.7, 3.25, and 3.26) or Abel's equation (subalgebras 3.5, 3.21, and 3.22). Subalgebra 3.20 specifies the Prandtl–Mayer wave. In this work, an autonomous system of subalgebra 3.2 is considered.

1. Submodel Equations. The family of subalgebras 3.2 is given by the basis of operators in the cylindrical system of coordinates $\{\partial_x, \partial_t, (\beta t + \alpha x)\partial_x + \alpha r \partial_r + \partial_\theta + \beta \partial_U\}$, where α and β are parameters of the family of subalgebras. By means of invariants, we define the representation of an invariant solution

$$U = \beta\theta + U_1(s), \qquad V = V(s), \qquad W = W(s), \qquad \rho = \rho(s), \qquad S = S(s),$$

$$s = r\,\mathrm{e}^{-\alpha\theta}, \qquad p = f(\rho, S),$$

where U is the velocity along the x axis, V and W are the radial and circular velocity components, ρ is the density, S is the entropy, p is the pressure, and the function f specifies the equation of state. For $\alpha \neq 0$, it is more convenient to use the invariant $\theta_1 = \theta - \alpha^{-1} \ln r$ instead of s.

The gas-dynamic equations yield four types of solutions.

1. $V = 0$, $\alpha \neq 0$, $W = 0$, and $p = p_0$ [$\rho = \rho(\theta_1)$ and $U_1 = U_1(\theta_1)$ are arbitrary functions]. This solution corresponds to an isobaric flow. The streamlines are parallel to the x axis. The level surfaces are cylinders with the generatrix parallel to the x axis, and with the directrix in the form of a logarithmic spiral. For the flow to be continuous, the function $\rho(\theta_1)$ should be periodic with a period 2π, and the function $U_1(\theta_1)$ should have a discontinuity $U_1(\theta_1 + 2\pi) = U_1(\theta_1) - 2\pi\beta$.

2. $V = 0$, $\alpha = 0$, and $W^2 = rp'\rho^{-1}$ [$p = p(r)$ ($p' \geqslant 0$), $\rho = \rho(r)$, and $U_1 = U_1(r)$ are arbitrary functions]. The world lines are determined by the equalities

$$x = \beta t^2 [p'(r_0)/(r_0\rho(r_0))]^{1/2}/2 + t(\beta\theta_0 + U_1(r_0)) + x_0,$$

$$r = r_0, \qquad \theta = t[p'(r_0)/(r_0\rho(r_0))]^{1/2} + \theta_0.$$

For $\beta = 0$, the trajectory is a helical line on a cylinder $r = r_0$ with a step $2\pi U(r_0) r_0 \rho(r_0)^{1/2} (p'(r_0))^{-1/2}$.

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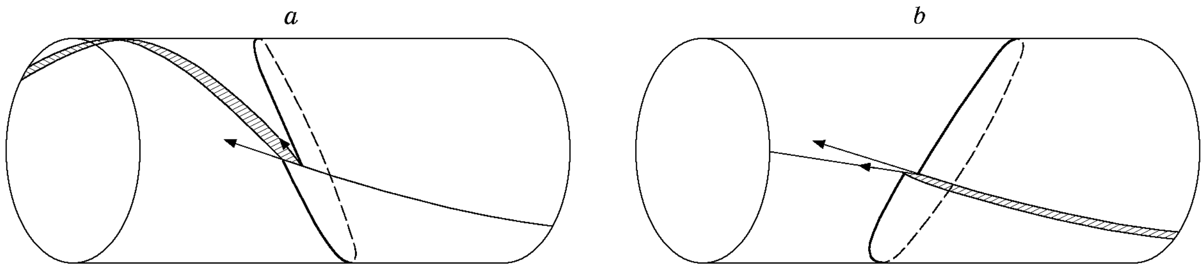

Fig. 1

3. $\alpha = 0$, $V \neq 0$, $S = S_0$, $r\rho V = E$, $rW = D$, $V^2 + 2i(\rho) = C^2 - D^2 r^{-2}$, and $U_1 = U_0 - \beta D C^{-1} \int \rho r^{-1}\, dr$ (S_0, E, D, U_0, and C are constants; $i = \int \rho^{-1}\, dp$ is the enthalpy). This solution corresponds to four vortex-free flows from a cylindric swirled source (for $E < 0$, from a sink) [5]. In such flows, there emerge limiting lines, on which acceleration turns into infinity. The flow is not extended beyond these lines.

4. $\alpha \neq 0$, $V \neq \alpha W$, $S = S_0$, $U_1 = \beta\alpha^{-1} \ln |\rho(V - \alpha W)| - \beta\theta_1$, and $V^2 + W^2 + 2i(\rho) = C^2$ [S_0 and C are constants]. The submodel of rank 1 is reduced to the autonomous equation

$$\frac{dV}{dW} = \frac{f_\rho(V + \alpha W) + W^2(V - \alpha W)}{f_\rho(W - \alpha V) - VW(V - \alpha W)} \tag{1}$$

and the quadrature

$$\alpha^{-1} \ln |W| + \int W^{-1}\, dV = \theta_1 - \theta_*, \tag{2}$$

where θ_* is a constant. For $\beta = 0$, the flow is vortex-free plane and is not a simple wave of submodels of rank 2 or 3 but a double wave of gas-dynamic equations.

2. Flow with a Helical Shock Wave. Two solutions of type 2 at $\beta = 0$ are matched via a shock wave. Let $h(x, r, \theta) = 0$ be the equation of the shock-wave surface. Then the shock-wave parameters can be determined: surface velocity in the normal direction $D_n = 0$, tangential component of the particle velocity $\boldsymbol{u}_\sigma = \boldsymbol{u} - u_n \boldsymbol{n}$, velocity projection onto the normal $u_n = \boldsymbol{u} \cdot \boldsymbol{n}$, and normal to the surface $\boldsymbol{n} = \nabla h |\nabla h|^{-1}$. From the condition on the shock wave $[\boldsymbol{u}_\sigma] = 0$, it follows that $h_r = 0$ and $h_x h_\theta^{-1} = [U](r[W])^{-1} = k = \text{const}$ ($[F] = F_2 - F_1$ is the difference of the parameter F values on different sides of the surface $h = 0$). Hence, we have $h = \theta + kx - \theta_0$, i. e., the trace of the shock-wave surface on the cylinder $r = r_0$ is a helical line with a step $-2\pi k^{-1}$.

The equations on a shock wave take the following form [6, p. 39]:

$$(krU_1 + W_1)^2 = (k^2 r^2 + 1)\rho_2 [p](\rho_1 [p])^{-1}, \quad (k^2 r^2 + 1)[W]^2 = [p][\rho](\rho_1 \rho_2)^{-1},$$

$$[U] = kr[W], \qquad H(p_2, \rho_2; p_1, \rho_1) = 0.$$

The second equality here specifies the Hugoniot adiabat, $W_1^2 = p_1'(r\rho_1)^{-1}$, and $W_2^2 = p_2'(r\rho_2)^{-1}$. If we specify k, ρ_1, and p_1 ($p_1' > 0$), the Hugoniot equation yields $\rho_2^{-1} = F(p_2, r) < \rho_1^{-1}$ for $p_2 > p_1$. Other equations determine U_1 and U_2, and in order to obtain $p_2(r)$, we have the ordinary differential equation

$$r(k^2 r^2 + 1) F p_2' = (((p_2 - p_1)(\rho_2^{-1} - F))^{1/2} \pm (p_1' \rho_1^{-1}(k^2 r^2 + 1))^{1/2} r^{1/2})^2.$$

Hence, $p_2' > 0$. The initial condition for this equation can be given arbitrarily: $p_2(r_0) = p_{20} > p_1(r_0)$.

Thus, in order to match two solutions via a helical shock wave with a specified step (the shock-wave trace on the cylinder $r = r_0$ is a helical line with a step independent of r_0), the solution ahead of the shock wave with arbitrary functions $\rho_1(r)$ and $p_1(r)$ ($p_1' > 0$) can be used. Then the functions $\rho_2(r)$, $p_2(r)$, $U_2(r)$, and $U_1(r)$ are obtained from the equations on the shock wave, and $p_2' > 0$.

Depending on the sign of $[W]$, there are two flow configurations: with a wall ahead of the shock wave (Fig. 1a) and with a wall behind the shock wave (Fig. 1b). The walls are formed by helical streamlines.

3. Integral Curves in a Hodograph Plane. In polar coordinates of the hodograph plane $V = Q\cos\vartheta$, $W = Q\sin\vartheta$, $Q^2 + 2\int\limits_0^\rho \rho^{-1} f_\rho\, d\rho = C^2$, submodel (1), (2) acquires the form

$$\frac{dQ}{d\theta_1} = -\frac{\alpha Q f_\rho}{Q^2(\cos\vartheta - \alpha\sin\vartheta)^2 - (\alpha^2+1) f_\rho}, \quad \frac{d\vartheta}{d\theta_1} = \alpha\,\frac{\alpha f_\rho + Q^2\sin\vartheta(\cos\vartheta - \alpha\sin\vartheta)}{Q^2(\cos\vartheta - \alpha\sin\vartheta)^2 - (\alpha^2+1) f_\rho}. \tag{3}$$

For the autonomous equation and quadrature, the resulting representations are

$$Q_\vartheta(\alpha f_\rho + Q^2\sin\vartheta(\cos\vartheta - \alpha\sin\vartheta)) + f_\rho Q = 0; \tag{4}$$

$$\alpha^{-1}(\ln|\sin\vartheta| + \ln Q) + \int Q^{-1}\cot\,\vartheta\, dQ = \theta_1 + \vartheta - \theta_*. \tag{5}$$

The physical integral curves of Eq. (1) or Eq. (4) lie within a circle $Q < C$. The circumference $Q = C$ is the integral curve of Eq. (1), on which $\rho = 0$ and $Cs|\sin\vartheta| = \exp(\alpha(\vartheta - \theta_*))$ ("vacuum" solution).

The streamlines are found from the equation

$$\frac{dr}{V} = \frac{r d\theta}{W} = \frac{\alpha\, dx}{\beta\ln|r\rho(V - \alpha W)|}.$$

The projections of streamlines onto the plane (r, θ) satisfy the equality

$$dr/r = \cot\,\vartheta\, d\vartheta \quad \text{or} \quad d\theta_1 = (1 - V/(\alpha W))\, d\theta. \tag{6}$$

The spatial flow is reconstructed from its projection onto the plane (r, θ) with nonintersecting projections of streamlines. If the projections intersect, a spatial flow is also possible for $\beta \neq 0$. Further, the projections of streamlines onto the plane (r, θ) are considered, which are conventionally called the streamlines.

For the "vacuum" solution, the equation for streamlines takes the form

$$(\alpha - \cot\,\vartheta)\, d(\theta + \vartheta) = 0.$$

For $\vartheta + \theta = \text{const}$, "vacuum" trajectories are straight lines $C|y\sin\theta_0 - z\cos\theta_0| = \exp(\alpha(\theta_0 - \theta_*))$ ($y = r\cos\theta$ and $z = r\sin\theta$). The position of the straight lines is determined by the parameter θ_0. For $\vartheta = \operatorname{arccot}\,\alpha^{-1}$, the "vacuum" trajectory is a logarithmic spiral $r = r_0\, e^{\alpha\theta}$ $[r_0 = C^{-1}\sqrt{1+\alpha^2}\exp(\alpha(\operatorname{arccot}\,\alpha^{-1} - \theta_*))]$, which is an envelope of the family of straight lines and a level line. In space, the level line corresponds to a cylindrical level surface, whose projection onto the plane $x = 0$ is a level line.

Equation (1) admits inversion $V \to -V$, $W \to -W$; consequently, the integral curves are symmetric about the origin of the coordinates.

There are five singularities:

— $V = W = 0$, $\rho = \rho_0$, $2i(\rho_0) = C^2$ are a focus and a stagnation point;

— $W = 0$, $V = \pm C$, $\rho = 0$ is a node for $\gamma \neq 2$ and a degenerate node for $\gamma = 2$ ($f(\rho) \sim \rho^\gamma$ as $\rho \to 0$);

— $V = \pm\alpha C(1+\alpha^2)^{-1/2}$, $\rho = 0$ is a saddle S.

For $\gamma \neq 2$, the integral curves enter the node $(C, 0)$, tangent to the straight line $W = \alpha(V - C)(\gamma-1)/(2-\gamma)$; the only integral line entering the saddle $[\alpha C(1+\alpha^2)^{-1/2}, C(1+\alpha^2)^{-1/2}]$ inside the circle $Q < C$ is tangent to the straight line (Fig. 2a)

$$((\gamma-1)\alpha^2 - \gamma)V + (2\gamma-1)\alpha W = C(\gamma-1)\alpha\sqrt{1+\alpha^2}. \tag{7}$$

For $\gamma = 2$, the integral lines enter the degenerate node $(C, 0)$, tangent to the circle $Q = C$ (Fig. 2b). The dashed curves in Fig. 2 are circles of critical velocities $Q = a_*$, $a_*^2 + I(a_*^2) = C^2$, $I(a^2) = 2i(\rho)$, $f_\rho = a^2$.

It follows from Eqs. (3) that the sign of $d\theta_1$ is determined by the sign of the expression $\Delta = Q^2(\cos\vartheta - \alpha\sin\vartheta)^2 - f_\rho(\alpha^2+1)$, $dQ < 0$ during the motion along the integral curve toward the center of the circle $Q < C$. On the curve $\Delta = 0$, the direction of the increase in θ_1 changes, and acceleration tends to infinity. This curve is a limiting line, beyond which no continuous flow is possible. On the limiting line, the following relation is fulfilled: $|\vartheta - \vartheta_0| = \varphi$ ($\cot\,\vartheta_0 = \alpha$, φ is the Mach angle, and $\sin^2\varphi = a^2 Q^{-2}$). Differentiation of the last equality with respect to ϑ yields $\sin 2\varphi = -(m + 2a^2 Q^{-2})Q^{-1}Q_\vartheta$, where $m = \rho f_{\rho\rho} f_\rho^{-1}$ and $a^2 = f_\rho$. Consequently, the limiting line is tangent to the circle at $\varphi = 0$, π, and $\pm\pi/2$. For $\varphi = 0$ we have $f_\rho = 0$, $\rho = 0$, $\vartheta = \vartheta_0$, and $Q = C$ (saddle). For $\varphi = \pi/2$, we have $f_\rho = Q^2$, $Q = a_*$, and $\vartheta = \vartheta_0 + \pi/2$ (tangency point of the circle of critical velocities). By virtue of the symmetry of integral curves with respect to the origin, the limiting line is an oval tangential to the circle of

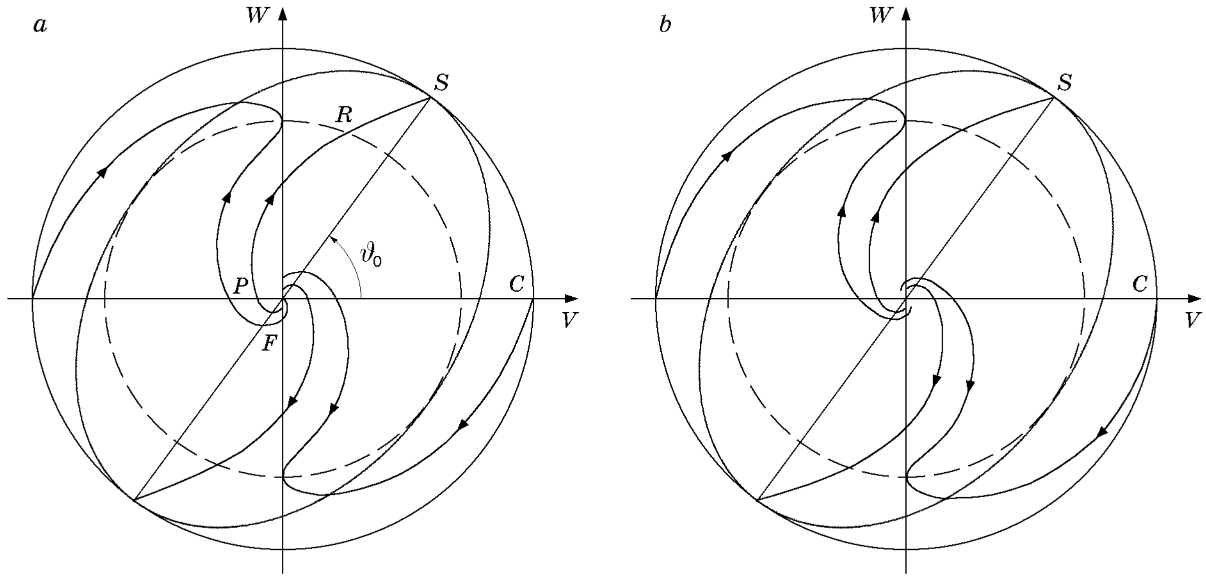

Fig. 2

limiting velocities $Q = C$ and to the circle of critical velocities. For a polytropic gas, the limiting line is an ellipse $Q^2 = C^2(\gamma - 1)(\gamma - \cos^2(\vartheta - \vartheta_0))^{-1}$ ($\gamma > 1$ is the ratio of specific heats).

For points in the vicinity of the circle $Q = C$, the condition $\Delta > 0$ is fulfilled. This implies that the value of θ_1 increases outside the oval (limiting line) during centripetal motion along the integral curve and decreases inside the oval. In Fig. 2, the arrows on the integral curves indicate the direction of increase or decrease in the absolute value of velocity Q as θ_1 increases.

4. Relative Position of Streamlines and Level Lines. The equations of level lines $\theta_1 = \theta - \alpha^{-1} \ln r = \theta_0$ or $r = r_0 \exp{(\alpha\theta)}$ specify logarithmic spirals. For $\beta = 0$, the level lines are within the range $0 \leqslant \theta_0 < 2\pi$ or $1 \leqslant r_0 < \exp{(2\pi\alpha)}$.

Assumption. *The streamline cannot be extended beyond the point of tangency with the level line.*

Proof. The level lines satisfy the equation $d\theta_1 = 0$. By using (6), we obtain the condition of streamline tangency to the level line

$$V = \alpha W, \quad \text{or} \quad \cot \vartheta = \alpha, \quad \text{or} \quad \vartheta = \vartheta_0.$$

For $\vartheta = \vartheta_0$, Eq. (3) yields the inequalities $\vartheta_{\theta_1} = -\alpha^2(1+\alpha^2)^{-1} < 0$ and $Q_{\theta_1} = \alpha Q(1+\alpha^2)^{-1} > 0$. Hence, on moving along the streamline toward $d\theta_1 > 0$ across the point of streamline tangency to the level line, the angle of inclination of the velocity vector to the vector of the position point monotonically decreases ($d\vartheta < 0$), and the absolute value of velocity monotonically increases ($dQ > 0$).

If the streamline lies on one side of the level line and is tangent to it, then the neighboring level line intersects this streamline at two points, where the values of ϑ are identical, which contradicts the monotonical change in the slope of the velocity vector to the position-point vector observed when the streamline passes through the tangency point.

Let the streamline cross the level line and be tangential to it at the intersection point. The angle between the point vector on the streamline and the streamline has an extremum at the tangency point, since the angle between the vector of any position point on the level line and the level line itself is constant and equal to ϑ_0, which contradicts the monotonical change in ϑ at the intersection of the tangency point.

It follows from the assumption that the integral curves of Eq. (1) should be treated only on one side of the straight line $V = \alpha W$ or $\vartheta = \vartheta_0$, $\vartheta = \pi + \vartheta_0$ (Fig. 2a). Every segment of the integral curve describes a continuous flow in the area bounded by two level lines.

5. Twisted Laval Nozzle. Let us consider an example of constructing a streamline for a segment of the integral curve SPF from the saddle to the focus up to the point F of intersection with the straight line $V = \alpha W$. The curve is specified by the equation $Q = Q(\vartheta)$ [$\vartheta \in (\vartheta_0, \vartheta_0 + \pi)$]. During the motion from the point S to the point F, the values of θ_1 decrease. As a streamline starting point, we take the point P, for which $\vartheta = \pi$, $Q = Q_P$,

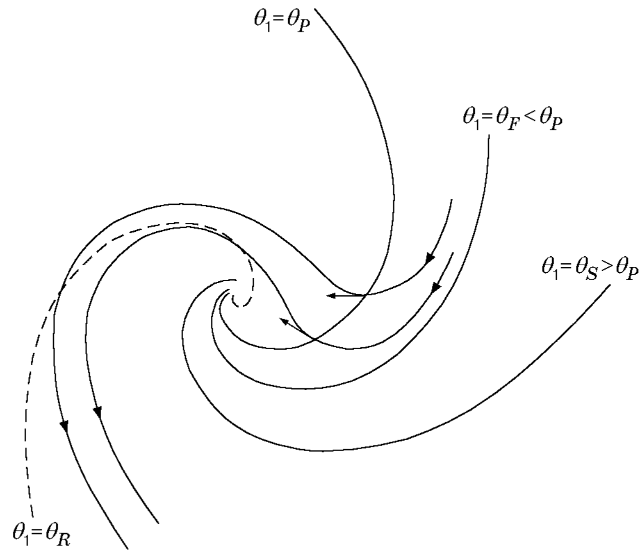

Fig. 3

and the velocity vector is directed to the center of the polar coordinate system in the plane of the flow. Let $\theta_0 = 0$ for this point; therefore, an initial ray of the polar coordinate system is specified in the plane of the flow. From (5) and (6), there follows that the streamline is determined by the equalities

$$\theta = \pi - \vartheta + \int_{\pi}^{\vartheta} \cot\,(\vartheta - \vartheta_0) Q^{-1}\,dQ, \qquad \theta_1 = \theta_* - \vartheta + \alpha^{-1} \ln Q + \int_{\pi}^{\vartheta} \cot\,\vartheta \Big(\frac{d\vartheta}{\alpha} + \frac{dQ}{Q}\Big), \tag{8}$$

where ϑ is a constant and the parameter θ_* defines the initial level line $\theta_1 = \theta_P = \theta_* - \pi + \alpha^{-1} \ln Q_P$ (Fig. 3).

For $\vartheta \in (\pi, \pi + \vartheta_0)$ Eq. (4) yields the inequality

$$-\alpha Q^{-1} Q_\vartheta = \{1 + Q^2(\tau - \alpha)/[\alpha f_\rho(1 + \tau^2)]\}^{-1} < 1,$$

where $\tau = \cot\,\vartheta \in (\alpha, \infty)$. Hence, by virtue of (8), there follows

$$\theta_1(\pi + \theta_0) = \theta_F = \theta_* - \pi - \vartheta_0 + \alpha^{-1} \ln Q_F$$

$$+ \int_{\pi}^{\pi+\vartheta_0} \cot\,\vartheta(\alpha^{-1} + Q^{-1}Q_\vartheta)\,d\vartheta > \theta_* - \pi - \vartheta_0 + \alpha^{-1} \ln Q_F,$$

$$\theta(\pi + \theta_0) = -\vartheta_0 + \int_{\pi}^{\pi+\vartheta_0} \cot\,(\vartheta - \vartheta_0) Q^{-1} Q_\vartheta\,d\vartheta > -\vartheta_0 - \alpha^{-1} \ln|\sin(\vartheta - \vartheta_0)|\Big|_{\pi}^{\pi+\vartheta_0} \to \infty.$$

Therefore, the streamline coils up from infinity (at infinity, the streamline approaches the level line $\theta_1 = \theta_F$).

To analyze the streamline behavior for the parameter ϑ varying over the range (ϑ_0, π), we use the pattern of integral curves (see Fig. 2) and equalities (4) and (7), from which there follows

$$Q_\vartheta\,|_{\vartheta=\vartheta_0} = \gamma C/(\alpha(1 - \gamma)), \qquad Q_\vartheta\,|_{\vartheta=\pi} = -\alpha^{-1} Q_P,$$

where $C = Q(\vartheta_0)$ and $Q_P = Q(\pi)$.

For the integral curve from the saddle, the following inequalities hold: $1 < -\alpha Q^{-1} Q_\vartheta < æ = \gamma(\gamma - 1)^{-1}$. Then, by virtue of (8), the estimates

$$\theta(\theta_0) = \pi - \vartheta_0 + \int_{\pi}^{\vartheta_0} \cot\,(\vartheta - \vartheta_0) Q^{-1} Q_\vartheta\,d\vartheta > \pi - \vartheta_0 - \alpha^{-1} æ \ln|\sin(\vartheta - \vartheta_0)|\Big|_{\pi}^{\vartheta_0} \to \infty,$$

$$\theta_1(\vartheta_0) = \theta_* - \vartheta_0 + \alpha^{-1} \ln C + \int_{\pi}^{\vartheta_0} \cot\,\vartheta(\alpha^{-1} + Q^{-1}Q_\vartheta)\,d\vartheta = \theta_S$$

are valid, where the integral converges, since the integrand cot $(\alpha^{-1} + Q^{-1}Q_\vartheta) \sim \alpha^{-2}a_P^{-2}Q_P^2$ has a finite limit at the singularity $\vartheta = \pi - 0$ or $\tau = \cot\vartheta \to -\infty$.

Thus, the streamline evolves to infinity as $\vartheta \to \vartheta_0$, approaching the level line $\theta_1 = \theta_S$.

Other streamlines are obtained by the transfer of velocity vectors along the level lines. Such a transfer can be performed infinitely far at infinity and unboundedly close to zero if all the level lines in the flow considered are different. This is possible if the following condition is fulfilled:

$$\theta_1(\vartheta_0) - \theta_1(\vartheta_0 + \pi) < 2\pi \quad \Rightarrow \quad \alpha^{-1}\ln\frac{C}{Q_F} + \int\limits_{\pi+\vartheta_0}^{\vartheta_0} \cot\vartheta(\alpha^{-1} + Q^{-1}Q_\vartheta)\,d\vartheta < \pi,$$

which can be satisfied by choosing a proper value of α. This case results in a swirled flow between spiral level lines with a transition through the velocity of sound ($\theta_1 = \theta_R$ is the sonic level line) from infinity to infinity with flow turning (Fig. 3).

Any streamline coils up from infinity, where it asymptotically approaches the level line $\theta_1 = \theta_F$. This streamline can be treated as a wall. At some convolution, the streamline turns, intersects the sonic line, then turns again and passes to infinity, asymptotically approaching the level line $\theta_1 = \theta_S$, which determines the vacuum region. For $\theta_S - \theta_F < 2\pi$, the resulting flow is univalent. For $\theta_S - \theta_F > 2\pi$, the continuous segment of the flow is possible only at one convolution, where the flow turns. There can also be a multivalent flow, where $\beta \neq 0$ and particles move perpendicular to the plane of variables (r, θ).

Two different streamlines specify an infinite twisted nozzle, which can be cut at every convolution (Fig. 3).

Conclusions. Invariant solutions of rank 1 for subalgebra 3.2 are represented as integrals or integral curves and a quadrature. This invariant submodel is integrable. Physical interpretation of the formulas obtained is a nontrivial problem. For every solution, it is necessary to identify the region of continuous gas flow, possible peculiarities, and asymptotic behavior at infinity. The physical pattern of gas motion can be supplemented if the possibility of matching invariant solutions via weak and strong discontinuities is defined. An example of shock-wave conjugation via a helical surface and an example of a continuous flow in a twisted Laval nozzle show the possibility of physical interpretation of invariant solutions.

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